Coordinate Bethe ansatz for the string *S*-matrix

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Abstract
We use the coordinate Bethe ansatz approach to derive the nested Bethe
equations corresponding to the recently found *S*-matrix for strings in $AdS_5 \times S^5$,
compatible with centrally extended $\mathfrak{su}(2|2)$ symmetry.

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1. Introduction

Recently, there has been a lot of progress in understanding the AdS/CFT correspondence.
One of the most important developments was the discovery of integrable structures on both
sides of the correspondence [1, 2]. Integrability provides new insights in how to calculate
spectra and how to study the correlation between the $AdS_5 \times S^5$ string sigma model and its
dual gauge theory. An important tool used to solve quantum integrable systems is a technique
called the Bethe ansatz. The Bethe ansatz has been applied to a variety of different problems
and there are two main variations known; the algebraic Bethe ansatz [3] and the coordinate
Bethe ansatz [4].

　　On the gauge theory side of the AdS/CFT correspondence, integrable structures emerged
via spin chains [2]; it was observed that conformal operators of $\mathcal{N} = 4$ SYM correspond
to eigenstates of an integrable spin chain at the planar one-loop level. Furthermore, the
scaling weights of the conformal operators coincide with energy eigenvalues of the spin chain
Hamiltonian. There is much evidence that integrability on the gauge theory side actually
extends to all loop order and the corresponding Bethe equations have been proposed for
certain asymptotic limits [5–7].

　　On the string theory side, integrability was exhibited for classical strings on $AdS_5 \times S^5$
[1]. One important open question is whether integrability is inherited by the quantum string.
Assuming that this is the case for the full quantum theory, a Bethe ansatz for the gauge-fixed
string sigma model was proposed [8]. The construction of [8] is based on the knowledge of
the finite-gap solutions of the classical string sigma-model [9]. The characteristic feature of
the quantum Bethe ansatz in comparison to the gauge theory Bethe ansatz is the appearance

of an additional scattering (dressing) phase constructed as a two-form on the vector space of local conserved charges. This dressing phase is universal and underlies the Bethe equations of the full-fledged sigma model [7]. However, in contradistinction to the gauge theory side, integrability at higher orders of string perturbation theory remains conjectural. For recent advances in this direction based on the direct world-sheet approach see [10–12].

The S-matrix describing the scattering of world-sheet excitations, respectively excitations of a certain spin chain, proved to be crucial in determining the relevant spectrum in the large volume (charge) limit [7–13]. This S-matrix turns out to be severely restricted if one imposes compatibility with the global symmetries of the model. It was first shown for the $\mathcal{N} = 4$ gauge theory that the relevant (super)algebra was centrally extended $\mathfrak{psu}(2|2) \oplus \mathfrak{psu}(2|2)$ [13]. The same algebra also emerges for superstrings on $AdS_5 \times S^5$ as a symmetry algebra for the light-cone gauge-fixed Hamiltonian [14]. It appears that both the two-particle S-matrix for superstrings on $AdS_5 \times S^5$ in the decompactifying limit and the S-matrix for the $\mathfrak{su}(2|2)$ dynamic spin chain [13] can be uniquely determined up to an overall phase factor by requiring invariance under this global symmetry algebra.

By imposing the requirement of crossing symmetry, which is a common property of relativistic field theories, one derives constraints on the dressing phase [15]. Based on earlier work [16–18], an explicit all-order perturbative expression of the dressing phase has been proposed for strings [19]. It agrees with the known string theory data and respects crossing-symmetry. The phase factor has also been proposed for the weakly-coupled $\mathcal{N} = 4$ gauge theory and further evidence was found that it is indeed related to the dressing factor from string theory by analytic continuation [20–22].

The two-particle S-matrix for superstrings on $AdS_5 \times S^5$ was recently determined using the symmetry invariance in [23]. This two-body S-matrix obeys the standard properties:

Yang–Baxter Equation $S_{23}S_{13}S_{12} = S_{12}S_{13}S_{23}$

Unitarity Condition $S_{12}(p_1, p_2)S_{21}(p_2, p_1) = \mathbb{I}$

Hermitian Analyticity $S_{21}(p_2, p_1)^\dagger = S_{12}(p_1, p_2)$

Crossing Symmetry $\mathcal{C}_1^{-1} S_{12}^{t_1}(p_1, p_2)\mathcal{C}_1 S_{12}(-p_1, p_2) = \mathbb{I}$,

where $\mathcal{C}$ is the charge conjugation matrix.

In general one encounters states with more than two excitations and hence one would also need a multi-particle S-matrix. However, the two-particle S-matrix contains all the relevant information if one assumes integrability. Scattering in integrable models preserves the number of particles and the set of their on-shell momenta [24]. In other words, there is no particle production and in the scattering process the particle momenta are merely exchanged. But, more importantly, these models admit factorization of the S-matrix, i.e. any multi-particle S-matrix, describing some scattering process, factorizes in a product of two-particle S-matrices. Note, nonetheless, that the string S-matrix we are considering, does not depend on the difference of rapidities, as is normally the case in relativistic two-dimensional integrable models possessing Lorentz symmetry. The factorized scattering is an extremely useful property, since it allows one to obtain the spectrum of a model from the two-particle S-matrix only.

Let us now explain how to derive the energy spectrum from the two particle S-matrix in the string theoretic picture and how the Bethe equations come into play. Consider creation operators $A_M^\dagger$ and annihilation operators A_M. The algebra these operators satisfy is the so-called Faddeev–Zamolodchikov (ZF) algebra [24, 25]:

$$A_1A_2 = S_{12}A_2A_1, \qquad A_1^\dagger A_2^\dagger = A_2^\dagger A_1^\dagger S_{12}, \qquad A_1A_2^\dagger = A_2^\dagger S_{12}A_1 + \delta_{12}, \qquad (1)$$

where S_{12} is the two-particle S-matrix and δ_{12} is the delta function depending on the difference of the momenta of the scattering particles. One can recognize that the standard properties of the two-particle S-matrix described above follow by requiring consistency of the ZF algebra relations. Asymptotic states are then constructed by acting with creation operators on the vacuum $|0\rangle$. A generic state, consisting of excitations with momentum p_i, will be of the form

$$A^{\dagger}_{M_1}(p_1) \dots A^{\dagger}_{M_N}(p_N)|0\rangle. \tag{2}$$

The Hamiltonian is given by one of the central charges of the symmetry algebra and hence, the dispersion relation is known [6]. From this, one can find the energy of such a state [6]:

$$E = \sum_{i=1}^{N} \sqrt{1 + 16g^2 \sin^2\left(\frac{1}{2}p_i\right)}. \tag{3}$$

This holds for any values of the momenta p_i. However, since we are dealing with closed strings, we have to impose periodicity on the wavefunction of world-sheet excitations. This requirement puts a restriction on the momenta in the form of a set of equations, usually referred to as (nested) Bethe equations. For the model in question this has been done recently by applying the algebraic Bethe ansatz approach [26].

In this paper, we will rederive the nested Bethe equations by using the coordinate Bethe ansatz in a way similar to [13]. First, we discuss the string S-matrix, present the equations obtained in [26] and briefly comment on how they were derived. Then we explain how the nested Bethe ansatz works, followed by a more detailed discussion on the involved calculations. We will also point out where the calculations differ from [13]. These results will be used to obtain the Bethe equations, which coincide with those found in [26][1]. We will also compare the obtained equations to those proposed in [7]. Finally, as a byproduct of our procedure we also obtain the explicit form of the Bethe wavefunction.

2. The S-matrix and algebraic Bethe ansatz

By demanding compatibility of the S-matrix describing world-sheet scattering, with centrally extended $\mathfrak{su}(2|2)$ symmetry, one can determine the S-matrix for strings on $AdS_5 \times S^5$ up to a phase factor [23]. We will consider this S-matrix:

$$\widehat{S}^{\mathrm{I}}_{12} = S^{\mathrm{string}}_{12}(p_1, p_2). \tag{4}$$

It acts according to the ZF algebra

$$\widehat{S}^{\mathrm{I}}_{12} \cdot A^{\dagger}_{M_1}(p_1)A^{\dagger}_{M_2}(p_2) = S^{N_1 N_2}_{M_1 M_2} A^{\dagger}_{N_2}(p_2)A^{\dagger}_{N_1}(p_1), \tag{5}$$

where the sum convention is used. Let us write the components of the S-matrix in the following way:

$$S^{\mathrm{I}}_{12}\big|A^{\dagger}_a(p_1)A^{\dagger}_b(p_2)\big\rangle = A\big|A^{\dagger}_{\{a}(p_2)A^{\dagger}_{b\}}(p_1)\big\rangle + B\big|A^{\dagger}_{[a}(p_2)A^{\dagger}_{b]}(p_1)\big\rangle + \tfrac{1}{2}C\epsilon_{ab}\epsilon^{\alpha\beta}\big|A^{\dagger}_\alpha(p_2)A^{\dagger}_\beta(p_1)\big\rangle$$

$$S^{\mathrm{I}}_{12}\big|A^{\dagger}_\alpha(p_1)A^{\dagger}_\beta(p_2)\big\rangle = D\big|A^{\dagger}_{\{\alpha}(p_2)A^{\dagger}_{\beta\}}(p_1)\big\rangle + E\big|A^{\dagger}_{[\alpha}(p_2)A^{\dagger}_{\beta]}(p_1)\big\rangle + \tfrac{1}{2}F\epsilon_{\alpha\beta}\epsilon^{ab}\big|A^{\dagger}_a(p_2)A^{\dagger}_b(p_1)\big\rangle$$

$$S^{\mathrm{I}}_{12}\big|A^{\dagger}_a(p_1)A^{\dagger}_\beta(p_2)\big\rangle = G\big|A^{\dagger}_\beta(p_2)A^{\dagger}_a(p_1)\big\rangle + H\big|A^{\dagger}_a(p_2)A^{\dagger}_\beta(p_1)\big\rangle$$

$$S^{\mathrm{I}}_{12}\big|A^{\dagger}_\alpha(p_1)A^{\dagger}_b(p_2)\big\rangle = K\big|A^{\dagger}_\alpha(p_2)A^{\dagger}_b(p_1)\big\rangle + L\big|A^{\dagger}_b(p_2)A^{\dagger}_\alpha(p_1)\big\rangle. \tag{6}$$

We will use the convention that the index $M = 1, 2, 3, 4$ runs through both bosonic and fermionic indices. The bosonic indices will be labelled $a, b = 1, 2$ and the fermionic indices

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[1] There is a subtle sign issue which is discussed in the following section.

will be labelled $\alpha, \beta = 3, 4$. The coefficients describing this scattering are easily seen from (5) to be

$$
\begin{aligned}
A &= a_1(p_1, p_2) & F &= 2a_7(p_1, p_2) \\
B &= -(a_1 + 2a_2)(p_1, p_2) & G &= a_5(p_1, p_2) \\
C &= 2a_8(p_1, p_2) & H &= a_{10}(p_1, p_2) \\
D &= a_3(p_1, p_2) & K &= a_9(p_1, p_2) \\
E &= -(a_3 + 2a_4)(p_1, p_2) & L &= a_6(p_1, p_2)
\end{aligned}
$$

The explicit form of the factors a_i is derived in [23] and is for convenience stated it in the appendix.

It is instructive to compare this S-matrix to that used in [13]. The S-matrix derived in [23], also describes the spin chain S-matrix, by making a particular choice for the coefficients a_i, which is given in the appendix. The relation of this spin chain S-matrix, with the S-matrix derived by Beisert, S^B, in [13], is given by complex conjugation

$$
S^B(p_1, p_2) = \overline{S}^{\text{chain}}(p_1, p_2), \tag{7}
$$

where S^{chain} is the aforementioned chain version of the S-matrix and we have chosen $\overline{x^\pm} = x^\mp$. This relation is more convenient in our case than the one given in [23]:

$$
S^B(p_1, p_2) = P\mathcal{P}S^{\text{chain}}(p_2, p_1)\mathcal{P}, \tag{8}
$$

where P and $\mathcal{P}$ are permutation and graded permutation respectively. In the latter case, the identification of the coefficients with A, B etc. is a little less straightforward.

By imposing periodicity on the discussed system, one derives restrictions on the momenta p_i. The algebraic version of the nested Bethe ansatz was recently applied [26] to derive the equations describing this. This is done by transforming the string S-matrix to Shastry's graded R-matrix, which makes that one can apply results earlier derived for the Hubbard model [27, 28]. From this, the Bethe equations for the string excitations are obtained and are given by

$$
\begin{aligned}
&\mathrm{e}^{\mathrm{i}p_k\left(-L+N-\frac{m_1^{(1)}}{2}-\frac{m_1^{(2)}}{2}\right)} = \mathrm{e}^{\mathrm{i}P} \prod_{i=1, i\neq k}^{N} S_0(p_k, p_i) \left[\frac{x_i^- - x_k^+}{x_i^+ - x_k^-}\right]^2 \prod_{\alpha=1}^{2} \prod_{j=1}^{m_1^{(\alpha)}} \left[\frac{y_j^{(\alpha)} - x_k^-}{y_j^{(\alpha)} - x_k^+}\right] \\
&\mathrm{e}^{\mathrm{i}\frac{P}{2}} \prod_{i=1}^{N} \left[\frac{y_j^{(\alpha)} - x_i^-}{y_j^{(\alpha)} - x_i^+}\right] = \prod_{l=1}^{m_2^{(\alpha)}} \left[\frac{v_j^{(\alpha)} - w_l^{(\alpha)} + \frac{i}{2g}}{v_j^{(\alpha)} - w_l^{(\alpha)} - \frac{i}{2g}}\right] \\
&\prod_{j=1}^{m_1^{(\alpha)}} \left[\frac{w_l^{(\alpha)} - v_j^{(\alpha)} + \frac{i}{2g}}{w_l^{(\alpha)} - v_j^{(\alpha)} - \frac{i}{2g}}\right] = \prod_{k=1, k\neq l}^{m_2^{(\alpha)}} \left[\frac{w_l^{(\alpha)} - w_k^{(\alpha)} + \frac{i}{g}}{w_l^{(\alpha)} - w_k^{(\alpha)} - \frac{i}{g}}\right],
\end{aligned} \tag{9}
$$

where y and w are auxiliary parameters and y and v are related via

$$
v = y + \frac{1}{y}. \tag{10}
$$

As one would expect, these equations are very similar to those describing the $\mathfrak{su}(2|2)$ dynamic spin chain [13], however, when comparing to [7], we see that there is a slight mismatch $L \leftrightarrow -L$, which is probably due the ambiguity in the Bethe ansatz as noted at the end of section 3 from [26]. In [13], the coordinate Bethe ansatz is used to derive the Bethe equations. In the following sections we will make the comparison explicit, i.e. we will rederive these equations by using the coordinate Bethe ansatz. However, we will set up our ansatz in such a way that the sign in front of L will coincide with that in [7].

Let us conclude this section by briefly discussing the difference between the algebraic and the coordinate Bethe ansatz approaches. In the algebraic case, one considers the monodromy matrix of the system. The starting point is to choose a particular state which is annihilated by the lower triangular part of the monodromy matrix. Then, from the upper triangular part, one can find creation operators. These operators are used to construct eigenstates of the trace of the monodromy (transfer matrix). From this construction one obtains the Bethe equations.

In the coordinate Bethe ansatz, one makes an ansatz for the wavefunction directly from the creation operators of the ZF algebra acting on a vacuum. Then one imposes periodicity on this wave function, which leads to the Bethe equations.

3. Procedure

In this section we will briefly discuss how the method of the coordinate Bethe ansatz will be applied here. Most of the calculational details will be treated in the following sections. The nested Bethe ansatz was first introduced in a seminal paper written by Yang [4]. We will mostly follow [7, 13].

The problem one wishes to solve is how to impose the periodicity condition on the wavefunction of the world-sheet excitations. This is needed since we are dealing with (non-interacting) closed strings of length described by a parameter L. Thus, the wavefunctions corresponding to world-sheet excitations should be L-periodic. The equations that capture this are called the Bethe equations.

Let us introduce some notation. The different asymptotic string states are built out of the ZF oscillators $A_i^\dagger(p_k)$ acting on a vacuum $|0\rangle$. Let us now consider the coordinate space with coordinates σ and suppose we create a state by using K^I creation operators.

Consider the case $\sigma_1 \gg \cdots \gg \sigma_{K^I}$. In this case, the excitations are far apart, which means that we neglect the interaction between them. Consider a creation operator $A_M^\dagger(\sigma)$, which creates a particle with the index M at position σ. By definition, the state $\left| A_{M_1}^\dagger(p_1) \ldots A_{M_{K^I}}^\dagger(p_{K^I}) \right\rangle := A_{M_1}^\dagger(p_1) \ldots A_{M_{K^I}}^\dagger(p_{K^I})|0\rangle$ describes K^I particles such that the particle with momentum p_i is to the left of the particle with momentum p_{i+1}. In other words, we have the identification

$$\left| A_{M_1}^\dagger(p_1) \cdots A_{M_{K^I}}^\dagger(\sigma_{K^I}) \right\rangle = \int_{\sigma_1 \gg \cdots \gg \sigma_{K^I}} \mathrm{d}\sigma_1 \ldots \mathrm{d}\sigma_{K^I} \, \mathrm{e}^{-\mathrm{i} \sum_{j=1}^{K^I} p_j \sigma_j} A_{M_1}^\dagger(\sigma_1) \ldots A_{M_{K^I}}^\dagger(\sigma_{K^I})|0\rangle.$$

(11)

The ansatz for the wavefunction in this sector is

$$\Phi(p_1, \ldots, p_{K^I}) = \chi_{M_1 \ldots M_{K^I}}(p_1, \ldots, p_{K^I}) \left| A_{M_1}^\dagger(p_1) \ldots A_{M_{K^I}}^\dagger(\sigma_{K^I}) \right\rangle, \quad (12)$$

where the indices are summed over. More generally, if Q is a permutation of the numbers $(1, \ldots, K^I)$, then in the sector where $\sigma_{Q_1} \gg \cdots \gg \sigma_{Q_{K^I}}$, we make a similar ansatz:

$$\Phi_Q(q_1, \ldots, q_{K^I}) = \chi_{Q;N_1 \ldots N_{K^I}}(q_1, \ldots, q_{K^I}) A_{N_1}^\dagger(q_1) \ldots A_{N_{K^I}}^\dagger(q_{K^I})|0\rangle. \quad (13)$$

Note that we can also just see this as a wavefunction in the sector $\sigma_1 \gg \cdots \gg \sigma_{K^I}$ with permuted momenta, by just a simple relabelling of the integration variables in (11). The region where there is interaction between the excitations links the different sectors and the relation between them is, by definition, given by the S-matrix. Thus, if we for example consider $Q = (12)$, then we obtain the following relation:

$$\chi_{12;N_1 N_2 \ldots}(p_2, p_1, \ldots, p_{K^I}) = S_{M_1 M_2}^{N_2 N_1} \chi_{M_1 M_2 \ldots}(p_1, p_2, \ldots, p_{K^I}). \quad (14)$$

More specifically, by the above relation, we can extend the asymptotic state in the region $\sigma_1 \gg \cdots \gg \sigma_{K^I}$ to the entire string in a unique way. The complete wavefunction for this asymptotic state will be given by

$$\chi_{M_1 \ldots M_{K^I}}(p_1, \ldots, p_{K^I}) \sum_{P \in S_{K^I}} \hat{S}_P \big| A^\dagger_{M_1}(p_1) \ldots A^\dagger_{M_1}(p_{K^I}) \big\rangle^I + \text{non-asymp}, \qquad (15)$$

where the sum runs over all permutations of $\{1, \ldots, K^I\}$. The periodicity condition is implemented by demanding that the wavefunction is invariant under

$$(\sigma_1, \ldots, \sigma_{K^I}) \to (\sigma_{K^I} + L, \sigma_1, \sigma_2, \ldots, \sigma_{K^I-1})$$
$$\to (\sigma_{K^I-k} + L, \ldots, \sigma_{K^I} + L, \sigma_1, \sigma_2, \ldots, \sigma_{K^I-k-1}) \qquad (16)$$

for all $k \in \{1, \ldots, K^I\}$. When we make ansatz (13), the periodicity condition (16), for any k, is given by

$$e^{-ip_k L} \hat{S}^I_{(k,-1)} \ldots \hat{S}^I_{(k,K^I)} \hat{S}^I_{(k,K^I)} \ldots \hat{S}^I_{(k,k+1)} \Phi(p) = \Phi(p), \qquad (17)$$

or written out explicitly

$$e^{-ip_K L} S^{\mu_1 \lambda_1}_{M_1 M_2}(p_k, p_{k-1}) S^{\mu_2 \lambda_2}_{\mu_1 M_3}(p_k, p_{k-2}) \ldots S^{\lambda_{K^I} \lambda_{K^I-1}}_{\mu_{K^I} M_{K^I}}(p_k, p_{k+1}) \chi_{M_1 \ldots M_{K^I}} = \chi_{\lambda_{K^I} \lambda_1 \ldots \lambda_{K^I-1}}. \qquad (18)$$

The term $S(p_k, p_k)$ is of course absent in the above product.

We are now left with solving this equation for the coefficients χ. This can be solved by making use of auxiliary systems that allow for additional Bethe ansätze. This is the nesting. Equation (18) can be seen as a matrix equation:

$$T^{\lambda_1 \ldots \lambda_{K^I}}_{k, M_1 \ldots M_{K^I}} \chi_{M_1 \ldots M_{K^I}} = \chi_{\lambda_{K^I} \lambda_1 \ldots \lambda_{K^I-1}}, \qquad (19)$$

The nesting procedure means that we will find the eigenvectors of the matrix operator T in steps. Finally, note that we have K^I of these equations, but from the Yang–Baxter equation it is easily verified that the different matrices T all commute and, hence, can be diagonalized simultaneously.

The idea is that we work in different steps or levels to diagonalize these matrices. This is done by considering auxiliary periodic systems. At each level we specify a 'new' vacuum and 'new' creation operators. The procedure is illustrated in figure 1. Each time a box indicates which operators are considered as creation operators and the operators without the box are considered background or vacuum. The end result is that we find the appropriate coefficient χ which solves equation (18) and hence we obtain the explicit wavefunctions of the system.

We start with the first level. The wavefunction for this level is given by the product of ZF generators:

$$\big| A^\dagger_{M_1}(p_1) \ldots A^\dagger_{M_{K^I}}(p_{K^I}) \big\rangle^I := \big| A^\dagger_{M_1}(p_1) \ldots A^\dagger_{M_{K^I}}(p_{K^I}) \big\rangle. \qquad (20)$$

In this level, we have K^I excitations or creation operators. Since we assume integrability, we know that this number is conserved.

For the next level, we define the first auxiliary system. This system is just a chain with K^I sites. One has to define what one considers as the vacuum state and what operators are to be considered as excitations. This is analogous to, for example, the Heisenberg spin chain where one can take all spins down to be the vacuum and one considers spins up as excitations. The choice made for the reference state at the second level is

$$|0\rangle^{II} = \big| A^\dagger_1(p_1) \ldots A^\dagger_1(p_{K^I}) \big\rangle^I, \qquad (21)$$

and all the other creation operators are considered to be creation operators on this new vacuum. This is shown in the second line of figure 1. In this section we exclude the $A^\dagger_2$ excitations
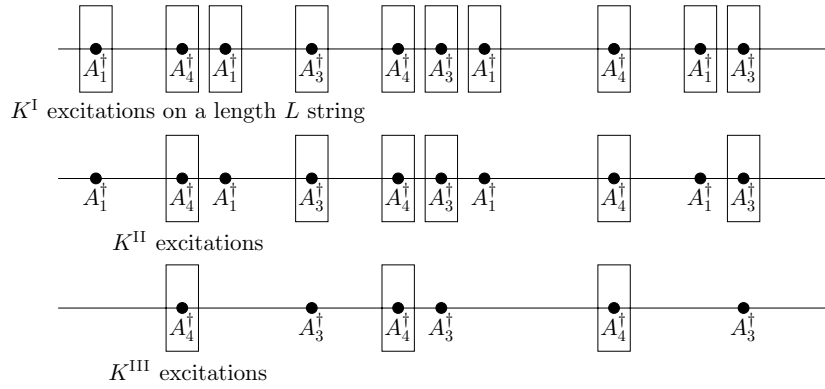
Figure 1. Schematic representation of the different levels of the nested Bethe ansatz. At each level, the plain dots represent the operators forming the vacuum and the boxes stand for the operators that are creation operators.

from the discussion since there is a subtle point about them which will be treated in the next section. However, for the understanding of the process, the absence of $A_2^\dagger$ plays no role. Now, one makes a second Bethe ansatz for this level. In this ansatz we will encounter additional parameters y, which will play the role of the momenta at this level. For one excitation, consisting of an $A_\alpha^\dagger$, the ansatz takes the form:

$$\left|A_\alpha^\dagger\right\rangle^{\mathrm{II}} = \sum_{k=1}^{K^{\mathrm{I}}} \Psi_k^{(1)}(y)\left|A_1^\dagger(p_1)\ldots A_\alpha^\dagger(p_k)\ldots A_1^\dagger(p_{K^{\mathrm{I}}})\right\rangle^{\mathrm{I}}. \tag{22}$$

This is just like a sum of plane waves. The way to determine the coefficients $\Psi_k(y)$ is to impose compatibility with the S-matrix, i.e.

$$S_{(k,l)}^{\mathrm{I}}\left|A_\alpha^\dagger\right\rangle^{\mathrm{II}} = S_{k,l}^{\mathrm{I,I}}(p_k, p_l)\left|A_\alpha^\dagger\right\rangle_{(k,l)}^{\mathrm{II}}, \tag{23}$$

where $\left|A_\alpha^\dagger\right\rangle_{(k,l)}^{\mathrm{II}}$ is $\left|A_\alpha^\dagger\right\rangle^{\mathrm{II}}$ with the momenta p_k and p_l interchanged and $S_{(k,l)}^{\mathrm{I,I}}(p_k, p_l)$ is a phase factor. This is a natural condition to impose, since this basically implies that the state, obtained in this way, is an eigenstate of the matrix T.

What the explicit form of $\Psi_k(y)$ is and how to deal with more than one excitation will be treated in the following section. However, the bottom line of this procedure is that we are one step closer to imposing periodicity and we now need to consider one creation operator less at the cost of introducing extra momenta y. We call the number of excitations at this level K^{II}, this can be interpreted as number of fermions in this system. In order to impose periodicity at this level, we again need to introduce an additional auxiliary system.

We proceed in a similar way and choose the reference state in the next level as

$$|0\rangle^{\mathrm{III}} = \left|A_3^\dagger \ldots A_3^\dagger\right\rangle^{\mathrm{II}} \tag{24}$$

and one only considers $A_4^\dagger$ as a creation operator. The Bethe ansatz made this time for a single excitation is of the same form as in the previous level:

$$\left|A_4^\dagger\right\rangle^{\mathrm{III}}(w) = \sum_{k=1}^{K^{\mathrm{II}}} \Psi_k^{(2)}(w)\left|A_3^\dagger(y_1)\ldots A_4^\dagger(y_k)\ldots A_3^\dagger(y_{K^{\mathrm{II}}})\right\rangle^{\mathrm{II}}. \tag{25}$$

One can now determine the coefficients by imposing compatibility with the level II S-matrix, which roughly describes the scattering of level II wavefunctions. The system is now reduced
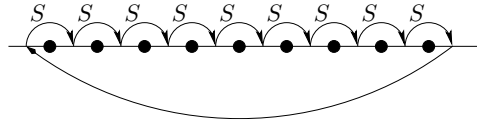
Figure 2. Schematic representation of the periodicity condition.

to just one type of creation operators of which there are K^{III}. This means that the wavefunction is fixed by giving the three different numbers of creation operators K^{I}, K^{II}, K^{III} and three sets of momenta, $\{p, y, w\}$.

By imposing the periodicity condition, one can derive the Bethe equations for the system. Note that periodicity is present in all three levels, which will give three sets of equations. The first one will, of course, corresponds to the solutions of the matrix equation (18). The other ones will put restrictions on the auxiliary momenta y, w. They will be derived in section 5.

The Bethe equations can be seen as scattering, via the relevant S-matrices, a creation operator around the string at the different levels. Each time one scatters two operators, the wavefunction picks up a phase factor. When the operator is back at its original position, the wavefunction should be unchanged (up to a phase factor). This is schematically depicted in figure 2. This amounts to the following Bethe equations:

$$e^{iL_{A,k}} = \prod_{B=\text{I}}^{\text{III}}\prod_{l=1}^{K^B} S^{BA}\big(x_l^B, x_k^A\big), \qquad (B,l) \neq (A,k) \qquad (26)$$

where A, B denote the different levels and, roughly, S^{AB} is the S-matrix describing how an excitation at the level B is scattered with an excitation of level A. Moreover, $e^{iL_{\text{I},k}} = e^{iLp_k}$ and $e^{iL_{\text{II},k}} = e^{iL_{\text{III},k}} = 1$ are phases depending on the level that is considered. This formula will be derived later on.

The phase e^{iLp_k} is dependent on the length of the string, L. When working in the uniform light-cone gauge [29, 30] one can express the length in terms of the conserved $U(1)$ charge J of the string: $L = J$.

4. Levels of the S-matrix

In this section we will derive the explicit form of the wave functions as well as the factors of the S-matrix corresponding to the different levels.

4.1. $S^{\text{I,I}}$

Recall that the level II reference state is given by $|0\rangle^{\text{II}} = \big|A_1^\dagger(p_1)\dots A_1^\dagger(p_{K^{\text{I}}})\big\rangle^{\text{I}}$. Since we assumed integrability, we only need to consider the action of a two particle S-matrix. As is easily seen from (7), the S-matrix acts trivially on the reference state at this level:

$$\begin{aligned}
\hat{S}_{(k,l)}^{\text{I}}|0\rangle^{\text{II}} =: S^{\text{I,I}}(p_k, p_l)|0\rangle_{(k,l)}^{\text{II}} \\
= A(p_k, p_l)|0\rangle_{(k,l)}^{\text{II}} \\
= S_0(p_k, p_l)\left[\frac{x_l^- - x_k^+}{x_l^+ - x_k^-}\right]\frac{e^{i\frac{p_l}{2}}}{e^{i\frac{p_k}{2}}}|0\rangle_{(k,l)}^{\text{II}}
\end{aligned} \qquad (27)$$

where (k, l) are the two particles that scatter and $|0\rangle^{\text{II}}_{(k,l)}$ is $|0\rangle^{\text{II}}$ with p_k and p_l interchanged. The factor S_0, the undetermined scalar phase of the S-matrix is given in the appendix.

4.2. Propagation and $S^{\text{II,I}}$

The next step is to consider excitations in this level. Let us start by considering a single excitation and see how this 'propagates' in the vacuum $|0\rangle^{\text{II}}$. From (7), it is easily seen that an insertion of $A_2^\dagger$, in a sea of $A_1^\dagger$ fields can 'decay' into the operators $A_3^\dagger$ and $A_4^\dagger$. Hence, the $A_2^\dagger$ behaves like a double excitation with respect to this reference state and we do not consider it here. A generic one-excitation state is now given by

$$\big|A_\alpha^\dagger\big\rangle^{\text{II}} = \sum_{k=1}^{K^{\text{I}}} \Psi_k^{(1)} \big|A_1^\dagger(p_1) \dots A_\alpha^\dagger(p_k) \dots A_1^\dagger(p_{K^{\text{I}}})\big\rangle^{\text{I}}. \tag{28}$$

The ansatz made for the coefficient is the following:

$$\Psi_k^{(1)} = f(p_k) \prod_{l=1}^{k-1} S^{\text{II,I}}(x_l). \tag{29}$$

The terms in the above expression can be viewed as a factor obtained by permuting the excitation with the background field ($S^{\text{II,I}}$), together with a factor for the combination of the excitation with the background field at position k ($f(p_k)$). As discussed in the previous section, one imposes compatibility with the level I S-matrix:

$$S^{\text{I}}_{(k,l)}\big|A_\alpha^\dagger\big\rangle^{\text{II}} = S^{\text{I,I}}_{k,l}(p_k, p_l)\big|A_\alpha^\dagger\big\rangle^{\text{II}}_{(k,l)}, \tag{30}$$

where $\big|A_\alpha^\dagger\big\rangle^{\text{II}}_{(k,l)}$ is $\big|A_\alpha^\dagger\big\rangle^{\text{II}}$ with the momenta p_k and p_l interchanged.

To explicitly solve the functions f and $S^{\text{II,I}}$, it is enough to consider a chain with only two sites:

$$\begin{aligned}
\big|A_\alpha^\dagger\big\rangle^{\text{II}} &= f(p_1)\big|A_\alpha^\dagger(p_1)A_1^\dagger(p_2)\big\rangle^{\text{I}} + f(p_2)S^{\text{II,I}}(p_1)\big|A_1^\dagger(p_1)A_\alpha^\dagger(p_2)\big\rangle^{\text{I}} \\
\big|A_\alpha^\dagger\big\rangle^{\text{II}}_{(1,2)} &= f(p_2)\big|A_\alpha^\dagger(p_2)A_1^\dagger(p_1)\big\rangle^{\text{I}} + f(p_1)S^{\text{II,I}}(p_2)\big|A_1^\dagger(p_2)A_\alpha^\dagger(p_1)\big\rangle^{\text{I}}.
\end{aligned} \tag{31}$$

Written out, the above compatibility condition gives the following equations:

$$\begin{aligned}
f(p_1)K + f(p_2)S^{\text{II,I}}(p_1)G &= f(p_2)A(x_1, x_2) \\
f(p_1)L + f(p_2)S^{\text{II,I}}(p_1)H &= f(p_1)S^{\text{II,I}}(p_2)A(x_1, x_2).
\end{aligned} \tag{32}$$

Now, one uses the first equation to solve for $S^{\text{II,I}}(p)$ in terms of $x^\pm$ and $f(p)$. This result can then be used, together with the second equation, to solve for $f(p_1)$ in terms of x_1^-, x_2^- and $f(p_2)$. By differentiating this expression with respect to x_2^-, one can solve for $f(p_2)$ only in terms of x_2^-. Doing this, one obtains

$$f(p_k) = \mathrm{e}^{-\mathrm{i}\frac{p_k}{2}}\frac{\eta(p_k)y}{y - x_k^-}, \tag{33}$$

where y is an integration constant. The constant y will play the role of a pseudo-momentum and we will explicitly include it in our notation from now on. The explicit solution for $S^{\text{II,I}}$ is easily obtained by using the above form of f. The complete solution is given by

$$\begin{aligned}
S^{\text{II,I}}(y, p_k) &= \mathrm{e}^{-\mathrm{i}\frac{p_k}{2}}\frac{y - x_k^+}{y - x_k^-} \\
f(y, p_k) &= \eta(p_k)\,\mathrm{e}^{-\mathrm{i}\frac{p_k}{2}}\frac{y}{y - x_k^-}.
\end{aligned} \tag{34}$$

4.3. Scattering and $S^{II,II}$

By using similar techniques, one also solves for the two excitation case. However, here one encounters additional degrees of freedom, which are dealt with by introducing an additional level of the S-matrix.

The natural ansatz to make for a two-excitation state is the superposition of two one-magnon states:

$$\left|A_\alpha^\dagger(y_1)A_\beta^\dagger(y_2)\right\rangle^{II} = \sum_{k<l=1}^{K^I} \Psi_k^{(1)}(y_1)\Psi_l^{(1)}(y_2)\left|A_1^\dagger(p_1)\ldots A_\alpha^\dagger(p_k)\ldots A_\beta^\dagger(p_l)\ldots A_1^\dagger(p_{K^1})\right\rangle^I.$$
(35)

It is easy to see that this solves the compatibility condition mentioned above if the two excitations are not neighbours. The additional freedom can be seen from the above formula. In this ansatz, we always have y_1 to the left of y_2, so this is very similar to the normal Bethe ansatz for a spin chain. In analogy to this, we introduce a second level S-matrix, S^{II}, which deals with interchanging y_1 and y_2. So a general two-excitation state $\left|A_\alpha^\dagger A_\beta^\dagger\right\rangle^{II}$, consisting of $A_\alpha^\dagger$ and $A_\beta^\dagger$, will be of the form:

$$\left|A_\alpha^\dagger A_\beta^\dagger\right\rangle^{II} = \left|A_\alpha^\dagger(y_1)A_\beta^\dagger(y_2)\right\rangle^{II} + S_{12}^{II}(y_1,y_2)\left|A_\alpha^\dagger(y_1)A_\beta^\dagger(y_2)\right\rangle^{II},$$
(36)

with

$$S_{12}^{II}(y_1,y_2)\left|A_\alpha^\dagger(y_1)A_\beta^\dagger(y_2)\right\rangle^{II} = M_{12}(y_1,y_2)\left|A_\alpha^\dagger(y_2)A_\beta^\dagger(y_1)\right\rangle^{II}$$
$$+ N_{12}(y_1,y_2)\left|A_\beta^\dagger(y_2)A_\alpha^\dagger(y_1)\right\rangle^{II}.$$
(37)

Indeed, S^{II} acts like a S-matrix with respect to the momenta y_i.

The $A_2^\dagger$ field has to be included as well. Since this behaves like a double excitation in the $A_1^\dagger$ background, we do not have the additional freedom corresponding to interchanging the y s. Instead, one includes an additional factor $f(y_1,y_2,p_k)$, which occurs when two excitations reside on the same position. This leads to the following ansatz:

$$\left|A_2^\dagger\right\rangle^{II} = \sum_{k=1}^{K^I} \Psi_k^{(1)}(y_1)\Psi_k^{(1)}(y_2)f(y_1,y_2,p_k)\left|A_1^\dagger(p_1)\ldots A_2^\dagger(p_k)\ldots A_1^\dagger(p_{K^1})\right\rangle^I.$$
(38)

To sum it all up, a general two excitation state is given by

$$|2\rangle^{II} = \left|A_\alpha^\dagger(y_1)A_\beta^\dagger(y_2)\right\rangle + \epsilon^{\alpha\beta}\left|A_2^\dagger\right\rangle + S_{12}^{II}\left|A_\alpha^\dagger(y_1)A_\beta^\dagger(y_2)\right\rangle.$$
(39)

The explicit form of the second level S-matrix and the two-excitation factor can be obtained by imposing compatibility with the level I S-matrix. When we explicitly write down the two-excitation state, we get

$$|2\rangle^{II} = f(y_1,p_1)f(y_2,p_2)S^{II,I}(y_2,p_1)\left|A_\alpha^\dagger(p_1)A_\beta^\dagger(p_2)\right\rangle^I$$
$$+ f(y_1,p_1)f(y_2,p_1)f(y_1,y_2,p_1)\epsilon^{\alpha\beta}\left|A_2^\dagger(p_1)A_1^\dagger(p_2)\right\rangle^I$$
$$+ f(y_1,p_2)f(y_2,p_2)S^{II,I}(y_1,p_1)S^{II,I}(y_2,p_1)f(y_1,y_2,p_2)\epsilon^{\alpha\beta}\left|A_1^\dagger(p_1)A_2^\dagger(p_2)\right\rangle^I$$
$$+ M(y_1,y_2)f(y_2,p_1)f(y_1,p_2)S^{II,I}(y_1,p_1)\left|A_\alpha^\dagger(p_1)A_\beta^\dagger(p_2)\right\rangle^I$$
$$+ N(y_1,y_2)f(y_2,p_1)f(y_1,p_2)S^{II,I}(y_1,p_1)\left|A_\beta^\dagger(p_1)A_\alpha^\dagger(p_2)\right\rangle^I.$$
(40)

When we write out the compatibility condition, we get four equations corresponding to the different configurations in (40). In order to make the equations not more cumbersome than they

already are, we introduce the short-hand notation $f_{kl} := f(y_k, p_l)$, $S_{kl} := S^{\mathrm{II},\mathrm{I}}(y_k, p_l)$, $M :=$ $M_{12}(y_1, y_2)$, $N := N_{12}(y_1, y_2)$. The equations coming from the configurations $\left|A_\alpha^\dagger A_\beta^\dagger\right\rangle$ and $\left|A_\beta^\dagger A_\alpha^\dagger\right\rangle$ are given by

$$\{f_{12}f_{21}S_{22} + Mf_{22}f_{11}S_{12}\}\Phi_{21}A = \{f_{11}f_{22}S_{21} + Mf_{21}f_{12}S_{11}\}\Phi_{12}\frac{D+E}{2}$$

$$+ Nf_{21}f_{12}S_{11}\Phi_{12}\frac{D-E}{2} + (-f_{11}f_{21}f_{121} + f_{12}f_{22}S_{11}S_{21}f_{122})\frac{C}{2} \tag{41}$$

and

$$Nf_{22}f_{11}S_{12}\Phi_{21}A = \{f_{11}f_{22}S_{21} + Mf_{21}f_{12}S_{11}\}\Phi_{12}\frac{D-E}{2}$$

$$+ Nf_{21}f_{12}S_{11}\Phi_{12}\frac{D+E}{2} - (-f_{11}f_{21}f_{121} + f_{12}f_{22}S_{11}S_{21}f_{122})\frac{C}{2}. \tag{42}$$

The equations coming from the double excitation $A_2^\dagger$ are easily seen to be given by

$$f_{11}f_{21}S_{12}S_{22}f_{121}A = \{f_{11}f_{22}S_{21} + (M-N)f_{21}f_{12}S_{11}\}\Phi_{12}\frac{F}{2}$$

$$+ f_{11}f_{21}f_{121}\frac{A-B}{2} + f_{12}f_{22}S_{11}S_{21}f_{122}\frac{A+B}{2} \tag{43}$$

and

$$f_{12}f_{22}f_{122}A = -\{f_{11}f_{22}S_{21} + (M-N)f_{21}f_{12}S_{11}\}\Phi_{12}\frac{F}{2}$$

$$+ f_{11}f_{21}f_{121}\frac{A+B}{2} + f_{12}f_{22}S_{11}S_{21}f_{122}\frac{A-B}{2}. \tag{44}$$

First, we add equations (41) and (42), which yields

$$M + N = -1. \tag{45}$$

From adding (43) and (44), it can be easily seen that $f(y_1, y_2, p_k)$ must be of the form

$$f(y_1, y_2, p_k) = \frac{\left(y_1 y_2 - x_k^+ x_k^-\right)}{\eta(p_k)^2}\frac{\left(x_k^+ - x_k^-\right)}{x_k^-}h(y_1, y_2). \tag{46}$$

Finally, this leaves us to determine $M - N$ and the factor $h(y_1, y_2)$. This is done by subtracting equations (41) and (42) and by subtraction of equations (43) and (44). The resulting equations can be solved analytically. First, one solves for $M - N$, by eliminating $h(y_1, y_2)$. This yields a fraction whose numerator and denominator are both polynomials in y_1, y_2, with coefficients depending on p_1, p_2. By doing a careful analysis of the numerator and denominator and by comparing the two, using the relation (A.2), one obtains the following result:

$$M - N = \frac{v_1 - v_2 + \frac{i}{g}}{v_1 - v_2 - \frac{i}{g}}, \tag{47}$$

where the new spectral parameter v_k is defined by

$$v_k := y_k + \frac{1}{y_k}. \tag{48}$$

For completeness we state the final solutions of M and N:

$$M = \frac{\frac{i}{g}}{v_1 - v_2 - \frac{i}{g}}$$

$$N = -\frac{v_1 - v_2}{v_1 - v_2 - \frac{i}{g}}. \tag{49}$$

The last thing to determine is the function h. Since we have an exact formula for M and N, finding the solution is rather straightforward.

$$h(y_1, y_2) = -\frac{i}{y_1 y_2} \frac{y_1 - y_2}{v_1 - v_2 - \frac{i}{g}}. \tag{50}$$

Let us stress that the solutions obtained are unique.

In [13] it is discussed how to generalize this to more than two excitations by using the supersymmetry generators. For this we need to consider the explicit four-dimensional representation of $\mathfrak{su}(2|2)$ and in particular the representation of the supersymmetry generators $\mathcal{Q}_\alpha$. We use the conventions from [23]

$$\mathcal{Q}_{k,\alpha}|0\rangle^{\mathrm{II}} = a_k \big| A_1^\dagger(p_1) \ldots A_\alpha^\dagger(p_k) \ldots A_1^\dagger(p_{K^{\mathrm{I}}}) \big\rangle^{\mathrm{I}}$$

$$\mathcal{Q}_{k,\alpha}\mathcal{Q}_{l,\beta}|0\rangle^{\mathrm{II}} = a_k a_l \big| A_1^\dagger(p_1) \ldots A_\alpha^\dagger(p_k) \ldots A_\beta^\dagger(p_l) \ldots A_1^\dagger(p_{K^{\mathrm{I}}}) \big\rangle^{\mathrm{I}} \tag{51}$$

$$\mathcal{Q}_{k,\alpha}\mathcal{Q}_{k,\beta}|0\rangle^{\mathrm{II}} = a_k b_k \epsilon^{\alpha\beta} \big| A_1^\dagger(p_1) \ldots A_2^\dagger(p_k) \ldots A_1^\dagger(p_{K^{\mathrm{I}}}) \big\rangle^{\mathrm{I}},$$

with $a_k = \sqrt{g}\sqrt{i(x_k^- - x_k^+)}\, e^{i\frac{p_{k+1}+\cdots+p_{K^{\mathrm{I}}}}{2}}$ and $b_k = -\frac{a_k}{x_k}$. We define dressed supersymmetry generators

$$\mathcal{Q}_{\alpha,k}^\pm := e^{-i\frac{P}{2}} \frac{x_k^\pm}{x_k^\pm - x_k^\mp} \mathcal{Q}_{\alpha,k}, \tag{52}$$

where $P := \sum_{i=1}^{K^{\mathrm{I}}} p_i$. By using the identity:

$$\frac{y}{y - x_k^-} = \frac{x_k^+}{x_k^+ - x_k^-} + \frac{x_k^-}{x_k^- - x_k^+} \frac{y - x_k^+}{y - x_k^-}, \tag{53}$$

we see that we can write the one excitation state as

$$\sum_{k=0}^{K^{\mathrm{I}}} \Phi_k\big(\mathcal{Q}_{\alpha,k}^- + \mathcal{Q}_{\alpha,k+1}^+\big)|0\rangle^{\mathrm{II}}, \qquad \Phi_k := \prod_{l=1}^{k} \frac{y - x_l^+}{y - x_l^-}. \tag{54}$$

This formula can be seen as a level II excitation which is moved through the vacuum via Φ_k, where it can be joined with the vacuum to the left by $\mathcal{Q}^-$ or to the right by $\mathcal{Q}^+$. The two-excitation state can now be written as

$$\big| A_\alpha^\dagger A_\beta^\dagger \big\rangle^{\mathrm{II}} = \frac{1}{2} \sum_{k=0}^{K^{\mathrm{I}}} \Phi_k(y_1)\Phi_k(y_2)\big\{ \mathcal{Q}_{\alpha,k}^-\mathcal{Q}_{\beta,k}^- + 2\mathcal{Q}_{\alpha,k}^-\mathcal{Q}_{\beta,k+1}^+ + \mathcal{Q}_{\alpha,k+1}^+\mathcal{Q}_{\beta,k+1}^+ \big\}|0\rangle^{\mathrm{II}}$$

$$+ \sum_{k<l=0}^{K^{\mathrm{I}}} \Phi_k(y_1)\Phi_l(y_2)\big(\mathcal{Q}_{\alpha,k}^- + \mathcal{Q}_{\alpha,k+1}^+\big)\big(\mathcal{Q}_{\beta,l}^- + \mathcal{Q}_{\beta,l+1}^+\big)|0\rangle^{\mathrm{II}}. \tag{55}$$

The first term is asymmetric since we need to make sure that y_1 stays to the left of y_2, so the first term can be seen as the ordered version of the second. The total two-excitation state is now given by

$$|2\rangle^{\mathrm{II}} = \big| A_\alpha^\dagger A_\beta^\dagger \big\rangle^{\mathrm{II}} + S_{12}^{\mathrm{II}}\big| A_\alpha^\dagger A_\beta^\dagger \big\rangle^{\mathrm{II}}, \tag{56}$$

with

$$S_{12}^{\mathrm{II}}\big| A_\alpha^\dagger(y_1)A_\beta^\dagger(y_2) \big\rangle^{\mathrm{II}} = M\big| A_\alpha^\dagger(y_2)A_\beta^\dagger(y_1) \big\rangle^{\mathrm{II}} + N S_{12}^{\mathrm{II}}\big| A_\beta^\dagger(y_2)A_\alpha^\dagger(y_1) \big\rangle^{\mathrm{II}}. \tag{57}$$

From this one can completely get rid of the explicit use of $A_2^\dagger$ in the formulae, since the corresponding factor has been distributed among the two different regions. This is now easily generalized to an arbitrary number of excitations.

Finally, by now introducing the third level vacuum (24), we can compute $S^{\text{II,II}}$. It follows that

$$S^{\text{II,II}} = -M - N = 1. \tag{58}$$

Note that we follow the convention of [13] and introduce and additional $-$ sign when we scatter two fermions.

4.4. Final levels

All that remains is a brief derivation of the last terms. The procedure is exactly the same as above, only the expressions involved are considerably more simple. We define the next reference state to be

$$|0\rangle^{\text{III}} = \left| A_3^\dagger(y_1) \ldots A_3^\dagger(y_{K^{\text{II}}}) \right\rangle^{\text{II}} \tag{59}$$

and we only need to consider the creation operators $A_4^\dagger$ as excitations (note that by the discussion at the end of the previous section, we tacitly split up the $A_2^\dagger$ into an $A_3^\dagger$ and an $A_4^\dagger$). Repeating the process described above, we are led to define a one-excitation state

$$\left| A_4^\dagger(w) \right\rangle^{\text{III}} = \sum_{k=1}^{K^{\text{II}}} \Psi_k^{(2)}(w) \left| A_3^\dagger(y_1) \ldots A_4^\dagger(y_k) \ldots A_3^\dagger(y_{K^{\text{II}}}) \right\rangle^{\text{II}}, \tag{60}$$

with

$$\Psi_k^{(2)}(w) = f^{(2)}(w, y_k) \prod_{l=1}^{k-1} S^{\text{III,II}}(w, y_k). \tag{61}$$

Note that, with a modest amount of foresight, we have already included the explicit dependence on the pseudo-momentum w, which will again come in as an integration constant. This time, the scattering relations are given by (37) and the compatibility relation is given by

$$S_{(k,l)}^{\text{II}} \left| A_4^\dagger(w) \right\rangle = S^{\text{II,II}} \left| A_4^\dagger(w) \right\rangle_{(k,l)}, \tag{62}$$

where this time the subscript (k, l) stands for interchanging y_k and y_l. This yields the equations:

$$
\begin{aligned}
M f^{(2)}(w, v_1) + N f^{(2)}(w, v_2) S^{\text{III,II}}(w, y_1) &= f^{(2)}(w, v_2) \\
N f^{(2)}(w, v_1) + M f^{(2)}(w, v_2) S^{\text{III,II}}(w, y_1) &= f^{(2)}(w, v_1) S^{\text{III,II}}(w, y_2),
\end{aligned}
\tag{63}
$$

in which the conventional extra $-$ is to be read in M, N and $S^{\text{II,II}}$. These equations are straightforwardly solved by

$$
\begin{aligned}
f^{(2)}(w, y_k) &= \frac{w - \frac{i}{2g}}{w - v_k - \frac{i}{2g}} \\
S^{\text{III,II}}(w, y_k) &= \frac{w - v_k + \frac{i}{2g}}{w - v_k - \frac{i}{2g}}.
\end{aligned}
\tag{64}
$$

When we consider a two excitation state, we will need to introduce the S-matrix, $S^{\text{III,III}}(w_1, w_2)$, which governs the interchanging of the w s, analogous to the previous level. The same ansatz for the two-excitation state as in the previous level can be made (without the term in which the excitations are on the same position, of course):

$$\left| A_4^\dagger(w_1) A_4^\dagger(w_2) \right\rangle^{\text{III}} = \sum_{l_1 < l_2} \Psi_{l_1}^{(2)}(w_1) \Psi_{l_2}^{(2)}(w_2) \left| \ldots A_4^\dagger(y_{l_1}) \ldots A_4^\dagger(y_{l_2}) \ldots \right\rangle^{\text{II}}. \tag{65}$$

By imposing the compatibility condition on the generic two excitation state at this level

$$\left|A_4^\dagger(w_1)A_4^\dagger(w_2)\right\rangle^{\mathrm{III}} + S^{\mathrm{III,III}}\left|A_4^\dagger(w_1)A_4^\dagger(w_2)\right\rangle^{\mathrm{III}} \tag{66}$$

one derives the following equation:

$$f_{11}^{(2)}f_{22}^{(2)}S_{21}^{\mathrm{III,II}} + S^{\mathrm{III,III}}f_{21}^{(2)}f_{12}^{(2)}S_{12}^{\mathrm{III,II}} = f_{12}^{(2)}f_{21}^{(2)}S_{22}^{\mathrm{III,II}} + S^{\mathrm{III,III}}f_{22}^{(2)}f_{11}^{(2)}S_{11}^{\mathrm{III,II}}. \tag{67}$$

This is solved by

$$S^{\mathrm{III,III}}(w_1, w_2) = \frac{w_1 - w_2 - \frac{i}{g}}{w_1 - w_2 + \frac{i}{g}}, \tag{68}$$

where again an additional–sign was absorbed. In general the K^{III}-excitation state is given by

$$\left|A_4^\dagger(w_1)\ldots A_4^\dagger(w_{K^{\mathrm{III}}})\right\rangle^{\mathrm{III}} + S^{\mathrm{III}} \cdot \left|A_4^\dagger(w_1)\ldots A_4^\dagger(w_{K^{\mathrm{III}}})\right\rangle^{\mathrm{III}}, \tag{69}$$

with

$$\left|A_4^\dagger(w_1)\ldots A_4^\dagger(w_{K^{\mathrm{III}}})\right\rangle^{\mathrm{III}}$$
$$= \sum_{l_1 < \cdots < l_{K^{\mathrm{III}}}} \Psi_{l_1}^{(2)}(w_1)\ldots\Psi_{l_{K^{\mathrm{III}}}}^{(2)}(w_{K^{\mathrm{III}}})\left|A_3^\dagger(y_1)\ldots A_4^\dagger(y_{l_1})\ldots A_4^\dagger(y_{l_{K^{\mathrm{III}}}})\ldots\right\rangle^{\mathrm{II}}. \tag{70}$$

Now the creation operator picture has completely dissolved and we are only left with the numbers K^{I}, K^{II}, K^{III} and the momenta p, y, w. From the above discussion, it is easily seen that

$$\begin{aligned}
K^{\mathrm{I}} &= N\left(A_1^\dagger\right) + N\left(A_2^\dagger\right) + N\left(A_3^\dagger\right) + N\left(A_4^\dagger\right) \\
K^{\mathrm{II}} &= 2N\left(A_2^\dagger\right) + N\left(A_3^\dagger\right) + N\left(A_4^\dagger\right) \\
K^{\mathrm{III}} &= N\left(A_2^\dagger\right) + N\left(A_4^\dagger\right),
\end{aligned} \tag{71}$$

where $N\left(A_M^\dagger\right)$ stands for the number of $A_M^\dagger$s in the state. K^{I} is the number of creation operators, K^{II} is the fermion number and K^{III} is the number of fermions of flavor $A_4^\dagger$. One can use these numbers in the above ansatz for the wavefunction and go back through all levels to obtain the total wavefunction of the system corresponding to p, y, w. Thus, one starts with a level III wave function with K^{III} excitations and by (69) one writes this as a linear combination of level II states, which can be written in terms of level I states.

4.5. Comparison with the spin chain picture

In [13], dynamic spin chains are considered. As discussed before, the S-matrix, is given by (7). When comparing the above discussion to that in [13], there are a few notational issues to have in mind. First, we have $A_a^\dagger \leftrightarrow \phi^a$ and $A_\alpha^\dagger \leftrightarrow \psi^\alpha$. Also, since in this case one deals with spin chains, there are also $\mathcal{Z}$ fields present in the discussion. This alters the first Bethe ansatz, in the sense that there is now a level I vacuum, consisting only of $\mathcal{Z}$s.

Other than this, the entire discussion basically goes through, apart from the fact that the spin chains are dynamic. This means that $\mathcal{Z}$ fields can be created and annihilated by creation operators $\mathcal{Z}^+$ and annihilation operators $\mathcal{Z}^-$. These operators give an additional phase factor in (40), which, in this picture, becomes

$$\begin{aligned}
\left|\Psi^{\mathrm{II}}\right\rangle = {}& f(y_1, x_1)f(y_2, x_2)S^{\mathrm{II,I}}(y_2, x_1)\left|\psi_1^\alpha\psi_2^\beta\right\rangle^{\mathrm{I}} \\
& + f(y_1, x_1)f(y_2, x_1)f(y_1, y_2)\frac{x_2^-}{x_2^+}\epsilon^{\alpha\beta}\left|\phi_1^2\phi_1^1\mathcal{Z}^+\right\rangle^{\mathrm{I}} \\
& + f(y_1, x_2)f(y_2, x_2)S^{\mathrm{II,I}}(y_1, x_1)S^{\mathrm{II,I}}(y_2, x_1)f(y_1, y_2)\epsilon^{\alpha\beta}\left|\phi_1^1\phi_1^2\mathcal{Z}^+\right\rangle^{\mathrm{I}}
\end{aligned}$$

$$+ M(y_1, y_2) f(y_2, x_1) f(y_1, x_2) S^{\text{II,I}}(y_1, x_1) \big| \psi_1^\alpha \psi_2^\beta \big\rangle^{\text{I}}$$
$$+ N(y_1, y_2) f(y_2, x_1) f(y_1, x_2) S^{\text{II,I}}(y_1, x_1) \big| \psi_1^\beta \psi_2^\alpha \big\rangle^{\text{I}}. \tag{72}$$

The results for the spin chain are given by

$$S^{\text{I,I}} = S_0(p_k, p_l) \frac{x_l^- - x_k^+}{x_l^+ - x_k^-} \qquad S^{\text{III,II}} = \frac{w - v_k + \frac{i}{2g}}{w - v_k - \frac{i}{2g}}$$

$$S^{\text{II,I}} = \frac{y - x_k^+}{y - x_k^-} \qquad\qquad S^{\text{III,III}} = \frac{w_1 - w_2 - \frac{i}{g}}{w_1 - w_2 + \frac{i}{g}} \tag{73}$$

$$S^{\text{II,II}} = 1.$$

Since the string and spin chain S-matrix only differ by x-dependent phase factors, one expects that the different levels of the S-matrix also only differ by phase factors, which is indeed the case. Furthermore, the factors depending only on the auxiliary parameters y, w coincide, which is not surprising since they are independent of x. Finally, note that $\mathcal{Z}$ fields basically give an extra level and hence one obtains in this case an extra Bethe equation, which corresponds to the level-matching condition.

5. Bethe equations

5.1. Final level

In this section, we will derive the Bethe equations, by imposing periodicity on the wavefunction. Consider a chain with K^{II} sectors and K^{III} excitations. The way to impose periodicity is depicted in figure 3.

In this figure the dots represent the K^{II} sites and the arrows represent the excitations at this level. The drawn configuration depicts one term of the ansatz for the wavefunction:

$$\big| A_4^\dagger(w_1) \dots A_4^\dagger(w_{K^{\text{III}}}) \big\rangle^{\text{III}} + S^{\text{III}} \big| A_4^\dagger(w_1) \dots A_4^\dagger(w_{K^{\text{III}}}) \big\rangle^{\text{III}}, \tag{74}$$

where $S^{\text{III}} \big| A_4^\dagger(w_1) \dots A_4^\dagger(w_{K^{\text{III}}}) \big\rangle^{\text{III}}$ stands for the different configurations corresponding to interchanging the ws. Furthermore, we have

$$\big| A_4^\dagger(w_1) \dots A_4^\dagger(w_{K^{\text{III}}}) \big\rangle^{\text{III}}$$
$$= \sum_{l_1 < \cdots < l_{K^{\text{III}}}} \Psi_{l_1}^{(2)}(w_1) \dots \Psi_{l_{K^{\text{III}}}}^{(2)}(w_{K^{\text{III}}}) \big| A_4^\dagger(y_1) \dots A_4^\dagger(y_{l_1}) \dots A_3^\dagger(y_{K^{\text{II}}}) \big\rangle^{\text{II}}. \tag{75}$$

Recall that the coefficients $\Psi_l^{(2)}$ are given by

$$\Psi_l^{(2)}(w) = f^{(2)}(v_l, w) \prod_l S^{\text{III,II}}(v_i, w), \tag{76}$$

but since we have periodicity, there is an ambiguity in choosing over which sites the product is taken. This is represented by the dotted lines in figure 3. The different depicted choices lead to a consistency check, corresponding to periodicity. Note that this is just the analogue of (16).

For each of the choices one can write down the explicit wavefunction, by just following the prescription given in the previous sections. We will give the explicit formulae for the first two lines, leaving the other ones for the interested reader.

The first configuration is just the superposition of K^{III} plane waves:

$$\sum_{l_1 < \cdots < l_{K^{\text{III}}}} \Psi_{l_1}^{(2)}(w_1) \dots \Psi_{l_{K^{\text{III}}}}^{(2)}(w_{K^{\text{III}}}) \big| A_4^\dagger(y_1) \dots A_4^\dagger(y_{l_1}) \dots A_3^\dagger(y_{K^{\text{II}}}) \big\rangle^{\text{II}}. \tag{77}$$
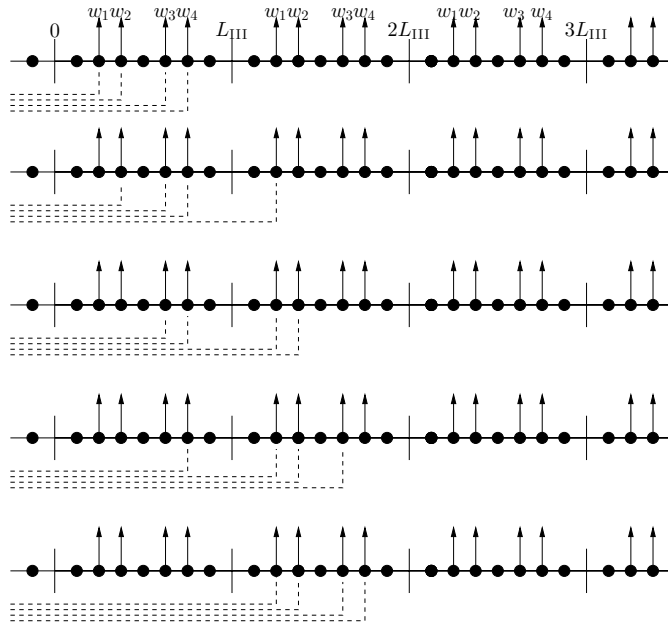
Figure 3. Schematic representation of the periodicity. The plain dots represent the operators forming the vacuum and the arrows stand for the excitations.

The second wavefunction is also a superposition of plane waves, but this time, the parameters w are in different order and we pick up additional factors of $S^{\text{III,II}}$:

$$S^{\text{III,II}}(w_1, v_1) \ldots S^{\text{III,II}}(w_1, v_{K^{\text{II}}}) \prod_{l \neq 1}^{K^{\text{III}}} S^{\text{III,III}}(w_1, w_l)$$

$$\times \sum_{l_1 < \ldots < l_{K^{\text{III}}}} \Psi_{l_1}^{(2)}(w_2) \ldots \Psi_{l_{K^{\text{III}}}}^{(2)}(w_1) \big| A_4^\dagger(y_1) \ldots A_4^\dagger(y_{l_1}) \ldots A_3^\dagger(y_{K^{\text{II}}}) \big\rangle^{\text{II}}. \tag{78}$$

The factors of $S^{\text{III,III}}(w_l, w_1)$ arise because w_1 is now to the right of the other w s. Since these two wavefunctions should be equal, it is easy to see that the following equation should hold:

$$\prod_{l=1}^{K^{\text{II}}} S^{\text{II,III}}(v_l, w_1) \prod_{l \neq 1}^{K^{\text{III}}} S^{\text{III,III}}(w_l, w_1) = 1, \tag{79}$$

where we define $S^{\text{II,III}}(v_l, w_k) := \frac{1}{S^{\text{III,II}}(w_k, v_l)}$. By considering the other choices one easily derives all the K^{III} Bethe equations for this level:

$$\prod_{l=1}^{K^{\text{II}}} S^{\text{II,III}}(v_l, w_k) \prod_{l \neq k}^{K^{\text{III}}} S^{\text{III,III}}(w_l, w_k) = 1. \tag{80}$$

We see that this coincides with (26). Finally, note that from these equations it also follows that the choice of origin is irrelevant as is seen by comparing the first and last line in figure 3, in which all the dotted lines are in the interval $[L, 2L]$ opposed to the interval $[0, L]$.

5.2. Other levels

In this section, we will only treat the second level Bethe equations. The level I Bethe equations are, of course, obtained in a similar way. We apply the same procedure as above. The only
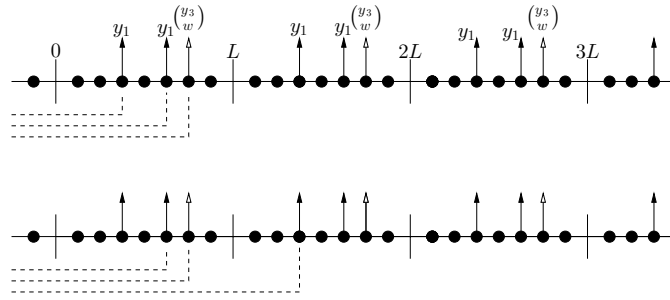
Figure 4. Schematic representation of the periodicity at the second level. The plain dots represent the operators forming the vacuum at this level. The black arrows stand for the A_3 excitations and the white arrow corresponds to the A_4 excitation.

difference is that we have more types of excitations. For ease of survey, we will only consider an explicit example, leaving the general case, which is not much more difficult, for the interested reader. We consider the case with two $A_3^\dagger$ and one $A_4^\dagger$ operator, see figure 4.

Again, the shown configurations in figure 4 only correspond to one term from the full wavefunction. Let us write the full wavefunction to make things more explicit. We have $K^{\mathrm{III}} = 1$, so our wavefunction at this level is

$$\left|A_4^\dagger\right\rangle^{\mathrm{III}}. \tag{81}$$

We can write this out in a linear combination of level II terms, as explained above:

$$\left|A_4^\dagger\right\rangle^{\mathrm{III}} = \Psi_1^{(2)}(w)\left|A_4^\dagger A_3^\dagger A_3^\dagger\right\rangle^{\mathrm{II}} + \Psi_2^{(2)}(w)\left|A_3^\dagger A_4^\dagger A_3^\dagger\right\rangle^{\mathrm{II}} + \Psi_3^{(2)}(w)\left|A_3^\dagger A_3^\dagger A_4^\dagger\right\rangle^{\mathrm{II}}. \tag{82}$$

Then we can do the same for each level II term and we obtain a large linear combination of level I wavefunctions,

$$\left|A_4^\dagger A_3^\dagger A_3^\dagger\right\rangle^{\mathrm{II}} = \sum_{l_1 < l_2 < l_3} \Psi_{l_1}(y_1)\Psi_{l_2}(y_2)\Psi_{l_3}(y_3)\left|\ldots A_4^\dagger \ldots A_3^\dagger \ldots A_3^\dagger \ldots\right\rangle^{\mathrm{I}} + \cdots \tag{83}$$

One such a term from this expansion is depicted in figure 4.

From the previous section, we know that the level III wavefunction can be written down unambiguously if the Bethe equations at that level are satisfied. However, for the level II wavefunction these equations still need to be derived. The procedure is completely analogous to that given above for $\Psi^{(2)}$ and the result is given by

$$\prod_{m=1}^{K^{\mathrm{I}}} S^{\mathrm{II,I}}(y_k, x_m) \prod_{l \neq k}^{K^{\mathrm{II}}} S^{\mathrm{II,II}}(y_k, y_l) \prod_{n=1}^{K^{\mathrm{III}}} S^{\mathrm{II,III}}(w_k, y_n) = 1, \tag{84}$$

which coincides with (26). Note that in the derivation of this equation, compatibility of the level III state under the level II S-matrix plays a crucial role. Finally, when comparing to (26), we use the definition

$$S^{AB}\left(x_k^A, x_l^B\right) = \frac{1}{S^{BA}\left(x_l^B, x_k^A\right)}. \tag{85}$$

The Bethe equation can be read as follows. We take the second line and we permute the level II excitation back to its original position. Doing this, we have to permute it past a level III excitation, giving a factor $S^{\mathrm{II,III}}(w_l, y_k)$, get it past level II excitations, which give $S^{\mathrm{II,II}}(y_l, y_k)$ and finally we have moved it through the vacuum, giving the $S^{\mathrm{II,I}}(y_k, x_l)$ terms. One can also read of the Bethe equations by explicitly writing out formula (18).

5.3. Results

Let us derive the explicit Bethe equations. We will first give them for the $\mathfrak{su}(2|2)$ case, from which the $\mathfrak{su}(2|2)^2$ follows.

From (26) we obtain the following three Bethe equations:

$$e^{ip_k(L+\frac{K^I}{2}-\frac{K^{II}}{2})} = e^{i\frac{P}{2}} \prod_{l=1,l\neq k}^{K^I} \left[S_0(p_k,p_l)\frac{x_k^+-x_l^-}{x_k^--x_l^+} \right] \prod_{l=1}^{K^{II}} \frac{x_k^- - y_l}{x_k^+ - y_l}$$

$$1 = e^{-i\frac{P}{2}} \prod_{l=1}^{K^I} \frac{y_k - x_l^+}{y_k - x_l^-} \prod_{l=1}^{K^{III}} \frac{y_k + \frac{1}{y_k} - w_l + \frac{i}{2g}}{y_k + \frac{1}{y_k} - w_l - \frac{i}{2g}} \tag{86}$$

$$1 = \prod_{l=1}^{K^{II}} \frac{w_k - y_l - \frac{1}{y_l} + \frac{i}{2g}}{w_k - y_l - \frac{1}{y_l} - \frac{i}{2g}} \prod_{l=1,l\neq k}^{K^{III}} \frac{w_k - w_l - \frac{i}{g}}{w_k - w_l + \frac{i}{g}}.$$

These equations coincide, as it should be, with those given in [26]. The equations for the full $\mathfrak{su}(2|2)^2$ case follow easily:

$$e^{ip_k(L+K^I-\frac{K^{II}_{(1)}}{2}-\frac{K^{II}_{(2)}}{2})} = e^{iP} \prod_{l=1,l\neq k}^{K^I} \left[S_0(p_k,p_l)\frac{x_k^+-x_l^-}{x_k^--x_l^+} \right]^2 \prod_{\alpha=1}^{2}\prod_{l=1}^{K^{II}_{(\alpha)}} \frac{x_k^- - y_l^{(\alpha)}}{x_k^+ - y_l^{(\alpha)}}$$

$$1 = e^{-i\frac{P}{2}} \prod_{l=1}^{K^I} \frac{y_k^{(\alpha)} - x_l^+}{y_k^{(\alpha)} - x_l^-} \prod_{l=1}^{K^{III}_{(\alpha)}} \frac{y_k^{(\alpha)} + \frac{1}{y_k^{(\alpha)}} - w_l^{(\alpha)} + \frac{i}{2g}}{y_k^{(\alpha)} + \frac{1}{y_k^{(\alpha)}} - w_l^{(\alpha)} - \frac{i}{2g}} \tag{87}$$

$$1 = \prod_{l=1}^{K^{II}_{(\alpha)}} \frac{w_k^{(\alpha)} - y_k^{(\alpha)} - \frac{1}{y_k^{(\alpha)}} + \frac{i}{2g}}{w_k^{(\alpha)} - y_k^{(\alpha)} - \frac{1}{y_k^{(\alpha)}} - \frac{i}{2g}} \prod_{l\neq k}^{K^{III}_{(\alpha)}} \frac{w_k^{(\alpha)} - w_l^{(\alpha)} - \frac{i}{g}}{w_k^{(\alpha)} - w_l^{\alpha} + \frac{i}{g}},$$

with $\alpha = 1, 2$.

Let us conclude by comparing these equations to those proposed in [7]. We compare our equations to the reformulated version of these equations in [10]. We see that they agree with the $(\eta_1, \eta_2) = (+, +)$ sector if one imposes the level matching condition $e^{iP} = 1$ and if one makes the following identifications:

$$\left(K^I, K^{II}_{(1)}, K^{II}_{(2)}, K^{III}_{(1)}, K^{III}_{(2)}\right) = (K_4, K_1 + K_3, K_5 + K_7, K_2, K_6)$$

$$\left(x_k^\pm; y_k^{(1)}; y_k^{(2)}; v_k^{(1)}; v_k^{(2)}; w_k^{(1)}; w_k^{(2)}\right) = \left(x_{4,k}^\pm; x_{3,k}; x_{5,k}; u_{3,k}; u_{5,k}; u_{2,k}; u_{6,k}\right), \tag{88}$$

with the parameter $L = J$.

6. Conclusions

In this note, the Bethe equations corresponding to the string S-matrix from [23] were derived by using the coordinate Bethe ansatz. The equations obtained for the string case correspond to those recently obtained in [26] and also coincide with the proposed equations in [7]. The method was already applied to the spin chain case in [13].

It would be interesting to study the relation between both approaches to the Bethe ansatz, especially the relation between the FZ creation operators and the creation operators obtained from the monodromy matrix in the algebraic Bethe ansatz.

There still remains a lot to be studied about these equations. The dependence on the momenta and total momentum P may yield interesting results, for example in the

thermodynamic limit. Furthermore, it would be interesting to see if the equations can be obtained with the method of [31].

Finally, since it is possible from the S-matrix from [23] to link the spin chain picture with the string picture, it would be nice to study if both cases can be linked via a continuous family of S-matrices and if the aforementioned procedure can be applied.

Acknowledgments

Appendix. *S*-matrix

For completeness, let us give the explicit form of coefficients of the S-matrix (7):

$$a_1(p_1, p_2) = S_0(p_1, p_2) \frac{x_2^- - x_1^+}{x_2^+ - x_1^-} \frac{\eta_1 \eta_2}{\widetilde{\eta}_1 \widetilde{\eta}_2}$$

$$a_2(p_1, p_2) = S_0(p_1, p_2) \frac{(x_1^- - x_1^+)(x_2^- - x_2^+)(x_2^- + x_1^+)}{(x_1^- - x_2^+)(x_1^- x_2^- - x_1^+ x_2^+)} \frac{\eta_1 \eta_2}{\widetilde{\eta}_1 \widetilde{\eta}_2}$$

$$a_3(p_1, p_2) = -S_0(p_1, p_2)$$

$$a_4(p_1, p_2) = S_0(p_1, p_2) \frac{(x_1^- - x_1^+)(x_2^- - x_2^+)(x_1^- + x_2^+)}{(x_1^- - x_2^+)(x_1^- x_2^- - x_1^+ x_2^+)}$$

$$a_5(p_1, p_2) = S_0(p_1, p_2) \frac{x_2^- - x_1^-}{x_2^+ - x_1^-} \frac{\eta_1}{\widetilde{\eta}_1}$$

$$a_6(p_1, p_2) = S_0(p_1, p_2) \frac{x_1^+ - x_2^+}{x_1^- - x_2^+} \frac{\eta_2}{\widetilde{\eta}_2} \tag{A.1}$$

$$a_7(p_1, p_2) = \mathrm{i} S_0(p_1, p_2) \frac{(x_1^- - x_1^+)(x_2^- - x_2^+)(x_1^+ - x_2^+)}{(x_1^- - x_2^+)(1 - x_1^- x_2^-) \widetilde{\eta}_1 \widetilde{\eta}_2}$$

$$a_8(p_1, p_2) = \mathrm{i} S_0(p_1, p_2) \frac{x_1^- x_2^- (x_1^+ - x_2^+) \eta_1 \eta_2}{x_1^+ x_2^+ (x_1^- - x_2^+)(1 - x_1^- x_2^-)}$$

$$a_9(p_1, p_2) = S_0(p_1, p_2) \frac{x_1^+ - x_1^-}{x_1^- - x_2^+} \frac{\eta_2}{\widetilde{\eta}_1}$$

$$a_{10}(p_1, p_2) = S_0(p_1, p_2) \frac{x_2^+ - x_2^-}{x_1^- - x_2^+} \frac{\eta_1}{\widetilde{\eta}_2}.$$

The parameters $x_k^{\pm}$ are related to the quasi-momentum of the magnons and the coupling constant in the standard way:

$$\frac{x_k^+}{x_k^-} = \mathrm{e}^{\mathrm{i} p_k}, \qquad x_k^+ + \frac{1}{x_k^+} - x_k^- - \frac{1}{x_k^-} = \frac{i}{g}. \tag{A.2}$$

Furthermore, the scalar function $S_0(p_1, p_2)$ is of the form:

$$S_0(p_1, p_2)^2 = \frac{x_2^+ - x_1^-}{x_2^- - x_1^+} \frac{1 - \frac{1}{x_1^+ x_2^-}}{1 - \frac{1}{x_1^- x_2^+}} \mathrm{e}^{\mathrm{i}\theta(p_1, p_2)} \mathrm{e}^{\mathrm{i} a(p_1 \epsilon_2 - p_2 \epsilon_1)}. \tag{A.3}$$

This *S*-matrix encompasses both the spin chain *S*-matrix and the string *S*-matrix, depending on the choice of parameters. To be precise, the spin chain *S*-matrix is given by making the following choice:

$$\eta_1 = \eta(p_1) \qquad \eta_2 = \eta(p_2)$$
$$\widetilde{\eta}_1 = \eta(p_1) \qquad \widetilde{\eta}_2 = \eta(p_2). \tag{A.4}$$

The string *S*-matrix is obtained by choosing:

$$\eta_1 = \eta(p_1)\, e^{i\frac{p_2}{2}} \qquad \eta_2 = \eta(p_2)$$
$$\widetilde{\eta}_1 = \eta(p_1) \qquad \widetilde{\eta}_2 = \eta(p_2)\, e^{i\frac{p_1}{2}}. \tag{A.5}$$

In both cases, $\eta(p_k) := \sqrt{i\left(x_k^- - x_k^+\right)}$.

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